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Modeling Experimental Time Series With Ordinary Differential Equations *

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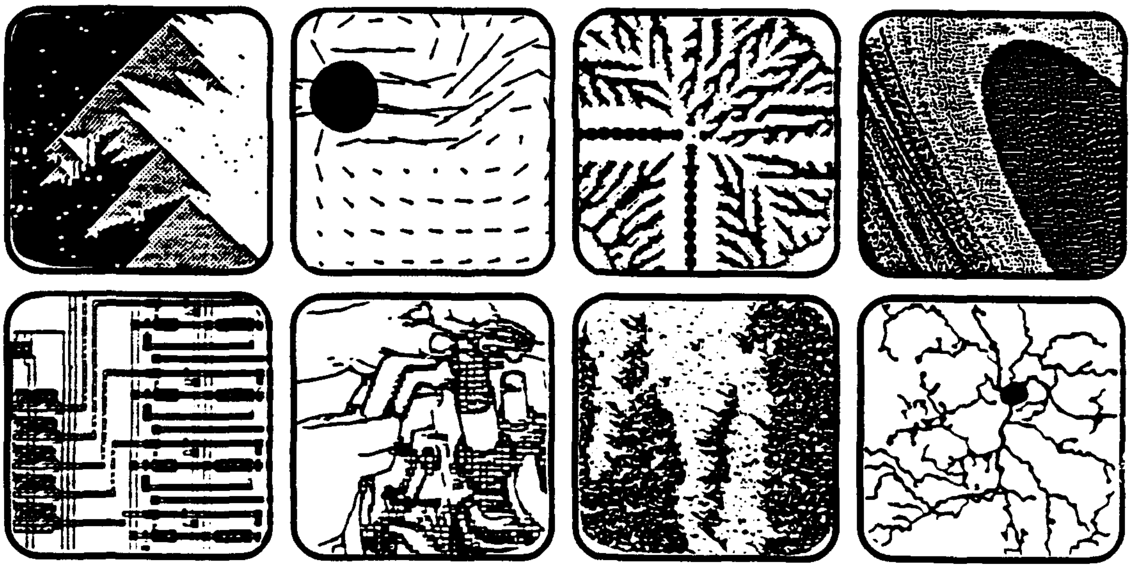

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Abstract. Recently some methods have been presented to extract ordinary differential equations (ODE) directly from an experimental time series. Here, we introduce a new method to find an ODE which models both the short time and the long time dynamics. The experimental data are represented in a state space and the corresponding flow vectors are approximated by polynomials of the state vector components. We apply these methods both to simulated data and experimental data from human limb movements, which like many other biological systems can exhibit limit cycle dynamics. In systems with only one oscillator there is excellent agreement between the limit cycling displayed by the experimental system and the reconstructed model, even if the data are very noisy. Furthermore we study systems of two coupled limit cycle oscillators. There, a reconstruction was only successful for data with a sufficently long transient trajectory and relatively low noise level.

1. Introduction

The modeling and analysis of dynamical systems is a field of increasing interest, in part because of applications in forecasting and control [?]. For a long time linear models have dominated descriptions of dynamical systems and control theoretic approaches [?]. Very complex and randomlike behavior was viewed from a statistical perspective in which very many degrees of freedom were involved. Only quite recently have nonlinear models emerged, capable of mirroring chaotic dynamics and other phenomena such as self excited oscillation [?]. Such systems can exhibit extremely complex dynamical behavior,

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even though the underlying dynamics may be low dimensional. On the other hand very high dimensional systems such as fluids or lasers can show simple and low dimensional dynamics, which may be described using low dimensional models [4,5]. In such systems most degrees of freedom are slaved and only a small number of order parameters are necessary for a description [5].

Along with new theoretical concepts have come a variety of different techniques to characterize the dynamics of such complex systems from any time series representative of the system's dynamical behavior. For example, generalized dimensions or Liapunov exponents have been used and calculations performed in complex systems all the way from the laser to the human EEG [6].

In this article, we present a new method for constructing a model equation from experimental time series that reproduces the dynamical behavior of the given system. The time series has to be represented in a low dimensional state space, e.g. position and velocity for a one dimensional oscillator. A trajectory of the system is given by the evolution of a point in the state space. A flow vector field, which is a function of the state vector components, governs this dynamical behavior. In order to model the given time series with a low dimensional ordinary differential equation (ODE) or flow vector, the dynamics has to be deterministic, i.e. no crossovers of the trajectory in the state space exist that are not due to noise [7]. Furthermore, the mapping of the given time series into state space has to be bijective. That is, a discrete point to point mapping must exist between the trajectory given as a time series, and the trajectory given in state space, and vice versa. In one of our experimental examples from human limb movement, pauses occur which do not follow this condition.

The reconstructed model equation, e.g. an ODE, is a "global description" [8], meaning the equation is valid on the whole subspace of the state space where experimental data are available. The model equation is represented in polynomials of the components of the state vector and the coefficients of these polynomials are fitted to the experimental time series. Differential equations of this type are always deterministic since they obey the Lipschitz condition [9]. An advantage of this closed model equation is that it can be compared to theoretical models and used to analyze and interpret the system's behavior. Furthermore, this description may be implemented to control a given system [10].

A disadvantage of a global description is that for higher dimensional systems and for higher order fits, the number of coefficients increases rapidly, rendering the approach cumbersome and unfavorable. In this regard we want to mention another approach, in which the dynamics is represented by a "local description" [2,8,11]. Here the trajectory is forecast by estimating the future trajectory by the trajectories of past states that come close to the present state. Thus, the model equations are only valid in a small area of state space, i.e., no closed description is possible. On the other hand, it may be possible to forecast higher dimensional systems better than using the global description.

The goal of the present work is to provide a global description for a given experimental time series, by reconstructing the underlying dynamics as well as possible. Of course the resulting model, in general, does not provide an explanation for the system's dynamics. Nevertheless, at least in simple cases, it may still be possible to connect the dynamics and the model equation to a deeper level of description, one which would allow for an understanding of the reconstructed equation. As an aside, such insight can be enhanced by empirical study. For example in [12] we show how an observable, the *relaxation time* (i.e. the time taken to return to the limit cycle after a perturbation), is related to the strength of the system's nonlinearity. In our approach only very obvious state vector components are used, namely the input data and their derivatives. Other state vectors, such as those that can be created using delays [13], are less helpful in analyzing and interpreting the reconstructed equations.

In earlier work the model equation was obtained by a direct fit of the flow vector field, using an approach developed by Cremers and Hübler [7]. A similar method was investigated in a slightly different manner by Crutchfield and McNamara [8]. The Cremers and Hübler approach, from now on referred to as the "flow method", produces very good results if data from a large enough subspace of the statespace are available. However, in many experimental applications the system is not chaotic; rather it is attracted very rapidly to a limit cycle. In such cases, only data from the limit cycle can be obtained, in the complete or near absence of transient data. The flow method usually does not perform well in the latter situation and often provides a model equation with unstable dynamics instead of a limit cycle. The advantage of the present "trajectory method", described in detail in section 2, is that it handles data on the limit cycle very well even when they are noisy; the procedure generally produces a stable fit.

It is worth noting that both the flow and the trajectory method can also be applied to chaotic time series with generally good results. In the present paper we study limit cycle oscillations, because there are many experimental systems in which this kind of behavior appears [14,15].

In section 3 we present some results, both for simulated data and for experimental data from human limb movements [16,12]. Data from biological systems are often very noisy, although it still proves possible in many cases, to describe the trajectories with low dimensional deterministic model equations [17]. First we study systems of single oscillators that may be described in a two dimensional state space. Since the experimental data are only on a limit cycle and are very noisy, the fitted coefficients sometimes depend quite strongly on parameters of the fit, such as the length of the included dataset. Nevertheless the trajectory of the given time series is approximated quite well. If additional information such as transient data are available we show that the reconstructed coefficients do not change very much.

In section 3 we also study systems of two coupled limit cycle oscillators, which constitute important models for biology in general [18] and in biological coordination in particular. There, self-organization is seen when different

coordination patterns arise as stable states of the coupled nonlinear dynamics [17,19,20]. Up to now, however, the reconstruction is shown to be successful for simulated data with moderate additional noise.

Section 4 is a short conclusion that raises future issues.

2. Reconstruction from time series and the trajectory method

As a first step it is necessary to chose a state space for the time series to be described. The dimension of this state space has to be at least as large as the dimension of the given time series. Various methods exist to calculate the dimension of an experimental time series [21].

In the systems investigated here, however, the dimension of the system is in most cases quite obvious (for actual calculation see Kay, Saltzman & Kelso [12]) and the physical meaning of the measured time series gives rise to the dimension of the used state space.

The state space can be constructed either with a time delay [13] or with derivatives. Differentiation increases noise, but this state space was used to describe the investigated low dimensional systems because only the first derivative in the trajectory method and the second derivative in the flow method are necessary. The advantage of using derivatives to define the state space is that the components have physical meaning. In many cases also the derivatives can be measured directly, thus reducing noise. Moreover, the state space components must be uncorrelated, which is often a problem in delay-constructed state spaces [8] if the delay is not chosen appropriately.

Many model equations of nonlinear oscillatory systems use a polynomial series as a favorable ansatz [7]. The flow vector field is represented by polynomials of the state vector components and the coefficients of these polynomials are fitted. Thus, given the d-dimensional state space vector

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \tag{2.1}$$

the i-th component of the flow is given by

$$\dot{x}_i = \sum_{i_1, i_2, \ldots, i_d = 0}^{m} c_{i_1 \ldots i_d} x_1^{i_1} \ldots x_d^{i_d} \tag{2.2}$$

where m is the highest order of the polynomials and $c_{i_1 \ldots i_d}$ are the fitted coefficients.

Although the method described in [7] fits the flow directly, in the trajectory method the information included in the trajectory, the evolution of the state vector in time, is used. Starting from appropriately chosen initial states $\vec{x}_e(t_j)$, the model equation is used to obtain an estimation of the states for later times $\vec{x}_m(t_j + \Delta t_l)$. The coefficients are fitted by minimizing the

distances between the states predicted by the model and the experimental states. The quality function Q is defined as

$$Q = \sum_{j,l=1}^{j_{max},l_{max}} \|\vec{x}_m(t_j + \Delta t_l) - \vec{x}_e(t_j + \Delta t_l)\| \tag{2.3}$$

where $\|\ \|$ is the Euclidian norm, $\vec{x}_e$ are the experimental states, t_j are the j_{max} times when the initial states are taken and Δt_l are the l_{max} times after which experimental and model states are compared. $\vec{x}_m$ is a function of the coefficients $c_{i_1 \ldots i_d}$ which are determined by the minimum of Q as a function of the coefficients:

$$Q_{min} = \min_{c_{i_1 \ldots i_d}} Q \tag{2.4}$$

The difference between this new trajectory method and the recent works of Cremers and Hübler [7] and Crutchfield and McNamara [8] is that it employs both short time and long time behavior for the fit. Experimental and model trajectory are compared at l_{max} different times Δt_l. The "flow method" uses only one small time step or only short time behavior, while the method by Crutchfield and McNamara uses only the long time behavior by using one large time step.

In eq.(2.3) the differences for all state space components are included. In most cases, however, only the squares of the differences between the measured experimental data and the corresponding model data

$$\tilde{Q} = \sum_{j,l=1}^{j_{max},l_{max}} \left(x_m(t_j + \Delta t_l) - x_e(t_j + \Delta t_l)\right)^2 \tag{2.5}$$

were used to define the quality function. In the experimental systems studied here, usually the first derivatives of the data are the other components of the state space, but these were not included in the minimized quality function. For such experimental data, the dynamics of the modelequation occasionally became unstable if the quality function Q (eq.2.3) was used, because the velocities were too noisy due to the differentiation process. On the other hand, when simulated data with a small amount of noise were used to test the present trajectory method, distances in the derivatives according to eq.(2.3) were also taken into account, though the quality of the fit was not changed. The goal of the method is, of course, to reconstruct only the experimental time series itself. It is possible to include more than one time series in the fitting process, e.g. in the coupled oscillator case. Also, if the experimental set up allows the derivative to be obtained directly, it might be favorable to also use these data.

In order to solve the minimization problem we used a routine provided by IMSL [22]. The calculation needs an initial model equation, i.e. initial coefficients of the model equation, which in most cases are chosen to be without any force. If the maximal $\Delta t_{l_{max}}$ is chosen very large, say about one oscillation period, another first guess for the initial coefficients is often necessary.

Otherwise problems with divergences appear. As a first guess, model equations that were obtained using smaller $\Delta t_{l_{max}}$ were used. A disadvantage of this method is the large amount of computing time necessary for large Δt_l. Therefore only in cases in which the flow method fails is this method favorable.

The Δt_l were chosen according to

$$\Delta t_l = \tau 2^{l-1} \tag{2.6}$$

where τ is the sampling rate and l_{max} was chosen in the range from 4 (8 samples) to 9 (256 samples). But also other definitions of Δt_l are possible, for example we used

$$\Delta t_l = 2l\tau \tag{2.7}$$

Nevertheless, no significant differences in the quality of the results were found.

The initial states $\vec{x}_e(t_j)$ should cover the whole limit cycle. However, not all experimental data can be used as initial states because the calculation time would bc too large. Thus a delay of b samples between two initial states was employed.

Since the trajectory method was applied only to single oscillators and to systems of two coupled oscillators, we restrict the problem to these systems. In the case of the single oscillator, the problem reduces to

$$\ddot{x} = \sum_{i_1, i_2 = 0}^{m} c_{i_1 i_2} x^{i_1} \dot{x}^{i_2} \tag{2.8}$$

and for $m = 3$ only 10 coefficients need be fitted.

Instead of integrating a model differential equation, it is also possible to use a map, which is much less time consuming. If very small time steps τ, smaller than 1% of the oscillation period are cnosen, the model equations for a single oscillator are given by

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{2}\tau\left(\dot{x}_n + \dot{x}_{n+1}\right) \\ \dot{x}_{n+1} &= \dot{x}_n + \sum_{i_1, i_2 = 0}^{m} a_{i_1 i_2} x_n^{i_1} \dot{x}_n^{\,i_2} \end{aligned} \tag{2.9}$$

The approximation for x_{n+1} is only valid for small time steps, however. For larger time steps the fit of a model equation for x_{n+1} is also necessary, thereby eliminating the advantage of reduced calculation time. Furthermore the model equations become more complicated, because more coefficients are necessary.

The coefficients for an ODE can simply be estimated from the coefficients for this map according to

$$c \approx \frac{a}{\tau} \tag{2.10}$$

if the time step is small enough.
In the case of two coupled oscillators the model equations are given by

$$\begin{aligned} \ddot{x}_1 &= \sum_{i_1,i_2,i_3,i_4=0}^{m} c_{1,i_1 i_2 i_3 i_4} x_1^{i_1} \dot{x}_1^{i_2} x_2^{i_3} \dot{x}_2^{i_4} \\ \ddot{x}_2 &= \sum_{i_1,i_2,i_3,i_4=0}^{m} c_{2,i_1 i_2 i_3 i_4} x_1^{i_1} \dot{x}_1^{i_2} x_2^{i_3} \dot{x}_2^{i_4} \end{aligned} \tag{2.11}$$

and for a fit up to third order 70 coefficients may be fitted.

3. Examples for the reconstruction of dynamical systems

In this section we demonstrate the reconstruction methods on simulated model equations as well as on real experimental data. Some of the latter were previously reported rhythmic movements of the single hand about the wrist [12] and a very lengthy single hand "control" time series obtained in a study of mode-locking by Kelso and DeGuzman [23]. Rhythmic behavior seems a logical place to start with the present approach because it is biologically meaningful in many species [14,20] .

3.1 Single oscillator, simulated time series

Simulated data for the well known van der Pol oscillator with the ODE

$$\ddot{x}_e = \dot{x}_e - \dot{x}_e x_e^2 - x_e \tag{3.1}$$

were studied as an example for a single oscillator with a limit cycle. The trajectory method was applied both to time series with and without transient data. Furthermore, an additional gaussian noise force was included in eq. (3.1) and again data with and without transient were fitted. In fig.(1 a) the used trajectory with noise is shown. In the following examples the order of the fitted polynomials was always 3.

For transient data without noise the coefficients were reproduced very well, with differences less than 1% relative to the coefficients of the simulated differential equation. The coefficients were not reproduced quite as exactly for data that lacked transients, but the differences were still quite small. The results of the fit both for model ODE and model maps are listed in Table 1.

The reason for the underestimation of the linear term in x and the overestimation of the linear term in $\dot{x}$ is not yet understood. One possibility is that the trajectory method may systematically overestimate the stability of a limit cycle, if the data do not include enough information about transient behavior. On the other hand, in many cases of fitted experimental and simulated data that lie on a limit cycle, the flow method produces a model with

unstable dynamics. This was the reason for the development of the trajectory method in the first place. Generally speaking, only very little information about the stabilizing terms is present on the limit cycle itself. Therefore, in the absence of transient data the method used often influences the stability of the fitted model.

The fit of the data with added noise also reproduces the coefficients quite well, if transient data are included, see Table 2. For a time series without transient data the coefficients of eq.(3.1) were not found and for different fit parameters the coefficients were also very different, see Table 3. But the trajectory of these model equations was always in good agreement with the trajectory of the van der Pol oscillator. Fig.1 (b) displays the trajectory for the model equation of Table 2 with $l_{max} = 7$. These results suggest that in many cases different model equations may be valid for the reconstruction of a given trajectory. However, if the information in the time series is larger, e.g., if transient information is provided, the variety of possible model equations can be drastically reduced and model coefficients obtained with different fit parameters are more or less equal.

3.2 Single oscillator, experimental time series

The foregoing conclusion applies also to experimental data obtained in studies of human rhythmical movement. As will be shown, the model coefficients can also be a function of the length of the time series included in the fit.

One time series was a 3Hz motion collected at a sampling frequency of 200Hz. The movement is very smooth and the limit cycle is not too noisy (see fig. 2 a). In this case the fit provided two different model equations, whose coefficients nearly always matched one of these equations. Also the standard deviations of the coefficients calculated with different fit parameters were fairly small. The first equation was found if only five oscillation periods were included in the data, while the other equation was found if more or less the whole time series of about ten periods was used. The two model equations are listed in Table 4; simulated trajectories are shown in fig.2 (b) and (c). Only a fit with Δt_l chosen according to eq.(2.7) gave a model equation, which was again different to the two model equations listed in Table 4. All these equations describe the given trajectory very well and not one of them can be considered to be better than the others. The rather high noise level often present in the experimental data make a decision about which model equation is favorable very difficult or even impossible.

Also for these experimental data the coefficients changed slightly and homogeneously with the maximal length of Δt_l. As mentioned earlier, this effect might be due to the reconstruction method itself. More theoretical work is required on this issue. The flow method also gave a model equation (see Table 4) with a stable dynamics. However, the time for the relaxation to the limit cycle is larger than for the trajectory method (see fig. 2 d). The reason for the difference in relaxation time lies in the different fit methods.

For experimental obtained at lower frequencies (e.g. 1.5 Hz) the fit be-

comes more problematic. In this case the subject pauses at maximal displacement (see fig.3a). Such pauses, which are dependent upon the frequency of motion, cannot be modeled with a two dimensional deterministic equation. As mentioned in the introduction the mapping of the time series into state space must be bijective, which is not the case for this time series. If one wants to also fit this trajectory another approach is necessary. Instead of fitting one ODE to this trajectory, it is possible to fit the movements in each direction independently with one ODE for each direction. The whole dynamics is then described with two discrete movements and a "trigger", which switches according to the performed frequency from one ODE to the other. Of course, we are not inferring from this description alone that this is the way the nervous system performs the task.

The fit of these discrete movements is not possible with the trajectory method as described in section 2, but a fit with the flow method is possible. However, the model equations are only valid in the corresponding part of the state space. If the initial conditions are not chosen appropriately, the trajectory may not relax to the limit cycle. Furthermore, the force must be switched off when the velocity passes zero, and the other ODE switched on sometime later. In fig.(3b,c) a trajectory of a model equation is shown, both as a time series and in state space. The trajectory is reconstructed quite well. A fit for different parts of the very long experimental time series was calculated. However, the coefficients differ, depending on the part of the time series that was used for reconstruction.

3.3 Reconstruction of coupled oscillators

For systems of two coupled limit cycle oscillators only simulated data have been reconstructed up to now. To understand the difficulties in the reconstruction of this system we first review briefly some results from studies of biological movements.

In the original experiments subjects were asked to perform oscillatory motions with the two index fingers. Only two phase-locked patterns prove to be stable: one in which the fingers move symmetrically (homologous muscles contracting in-phase) and the other in which the fingers alternate (homologous muscles contracting anti-phase). The discovery, later examined in great detail and shown to take the form of a nonequilibrium phase transition (e.g. enhanced fluctuations, critical slowing down of the order parameter were all observed) was that the anti-phase pattern lost stability at a critical movement frequency and shifted to the more stable in-phase, symmetric pattern (see [20] for reviews). Using concepts of synergetics, the order parameter dynamics for relative phase were identified and later derived by nonlinear coupling [17] The significance of coupled nonlinear oscillators for biology in general [18] and the foregoing results showing that the stability and change of coordinated movement behavior may be understood as nonequilibrium phase transitions, make systems of nonlinearly coupled oscillators a natural choice of study for the present reconstruction methods. We limit our analysis to

simulated data for the moment.

For a system of equal oscillators in a phase locked state the positions and the velocities of the oscillators are strictly correlated, even equal for the in-phase movement. Since it is necessary to have uncorrelated state vector components, a fit with the state vector $\vec{x} = (x_1, \dot{x}_1, x_2, \dot{x}_2)$ is not possible with the present reconstruction methods. However, since the positions and velocities are strictly correlated, a model equation assuming two independent oscillators describes the given time series as well as a system of coupled oscillators. Such a model may even be considered more favorable, since many fewer coefficients are necessary to model single oscillators without coupling than a full system of coupled oscillators.

To reconstruct the coupling terms transient data must be available. If the system is attracted to a phase-locked state that is in-phase, the state vector

$$\vec{x} = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_1 - x_2 \\ \dot{x}_1 - \dot{x}_2 \end{pmatrix} \tag{3.2}$$

is appropriate, since the third and fourth components are attracted to zero and the transient to the phase-locked state is easily visualized. The differential equation of one studied example is given by

$$\ddot{x}_1 = -2.0x_1 + 0.2\dot{x}_1 - 0.4x_1^2\dot{x}_1 + 0.1x_2 + 0.3\dot{x}_2 \tag{3.3}$$

$$\ddot{x}_2 = -2.0x_2 + 0.2\dot{x}_2 - 0.4x_2^2\dot{x}_2 + 0.1x_1 + 0.3\dot{x}_1 \tag{3.4}$$

An example of a simulated trajectory with additional noise is shown in fig.4a and b together with the trajectory of the fitted model with the same initial conditions. These simulations may be compared with the simulations in [17] and especially [24], which do not, of course, reconstruct the full trajectory. For the fit, both the flow and the trajectory method were used and both reconstruct the trajectory very well. Notably, in systems with a shorter transient a good reconstruction is also possible. Whether a reconstruction is possible or not depends again on the length of the available transient and the noise in the experimental data.

The coefficients of the simulated differential equation are only reconstructed when the additional noise is smaller than in the plotted example. A related problem appears when the model is simulated using other initial conditions. If the simulated time series is too noisy or the transient too short, the model equation will often produce an unstable trajectory. The reason is that the flow vector field is reconstructed well in the subspace for which experimental or simulated data are available, but poorly outside this region. For the systems of coupled limit cycle oscillators studied here, only data on the limit cycle are available, and the model equation is valid only close to the

limit cycle. In two dimensional systems with a single oscillator this problem is not so important, although divergences appear when the initial conditions are set further away fom the limit cycle. In higher dimensional systems such issues are much more important.

Finally we want to mention that a reconstruction of the dynamics of coupled oscillators is possible also for other systems that are not phase-locked. There the results are often better since a larger subspace of the state space is reached.

4. Conclusion

In a number of cases (both simulated and experimental) we have shown that it is possible to reconstruct the dynamics of a system from an experimental time series using a novel method, the trajectory method. With reconstructing methods the information included in the time series is reduced to a very small number of coefficients of an ODE, including the same information as the time series. On the other hand, we have drawn attention to the drawbacks of such a reconstruction technique as well as its advantages in comparison to other methods in literature. A number of questions remain that are worth addressing.

One concerns how the amount of information that is necessary to produce a successful fit can be estimated and measured. This is a crucial issue: As mentioned above, fitted model equations generally cannot provide the relaxation time to the limit cycle if no transient data are available. How much information must be included in the time series, so that predictions for global measures such as the relaxation time are possible?

A further issue concerns the influence of the used fit method and the used fit parameters on the results both of which need to be studied systematically. Finally, the fit method should give a minimal model equation for the experimental time series, where minimality is to be measured in terms of order and dimensionality of the fit and the quality of the description of the time series.

5. Survey

The essence of complex biological systems that possess many degrees of freedom is that they can temporarily assemble their components into a much lower dimensional structure whose dynamics are nonlinear and hence capable of exhibiting complicated behavior, among which are oscillations, bifurcations, multistability, even deterministic chaos. Thus the brain, with 10^{14} *neurons and neuronal connections,* $\sim 10^3$ *cell types, hundreds of active chemicals is a materially complex system, whose dynamics nevertheless can often be demonstrated to be much lower dimensional (for review see e.g. [6]). Similarly, the motor system (ignoring for the moment its neural and vascular support processes) has minimally 792 muscles and 100 joints to coordinate during the course of typical activities. Watch a concert pianist, for example, and you will see a continuous blending of posture, expression, and articulated biman-*

ual performance. Like music itself, the artist's performance is coordinated on many time scales.

To guarantee stability of behavior on the one hand and flexibility on the other, the behavioral dynamics must be nonlinear. In fact, in the last few years all the signature features of nonequilibrium phase transitions, i.e. where patterns form in a spontaneous, self-organized fashion under non-specific changes in control parameters [5] have been observed and modeled in human movement coordination tasks (e.g. [16,17,19,20]). Once the macroscopic behavioral patterns and their dynamics have been found (through the methodology of phase transitions or qualitative changes in behavior), they can be derived or synthesized by cooperatively coupling nonlinear oscillators (e.g. [17,24]). Such cooperative coupling affords both a rigorous definition of the relation between levels of description and indicates how the compression of degrees of freedom occurs, e.g. from the limit cycle dynamics of the individual components to the point attractor dynamics of the collective variable, or order parameter relative phase. Thus, the reduction is not to some fundamental unit or level of analysis, but rather to find laws at one level and derive them from another.

In this regard, it is quite clear that nonlinear oscillators play quite a central role at many levels of description and throughout the natural sciences. A large variety of complex systems effectively reduce their degrees of freedom to the frequency and phase or mode-locking dynamics of coupled nonlinear oscillators ([14] for recent review). One of us is inclined to think that this is a strategy for the nervous system too which must perform tasks that require the coordination among many spatially and temporally distributed processes (see e.g. [23]). Such a coordinative design is not only apparent in voluntary behavior but in many other "systems", ranging from so-called locomotory-respiratory rhythms, cardiac rhythms, cardiovascular-respiratory-somatomotor interactions, to individual neurons such as the squid axon and so forth. Stationary states, oscillations, bifurcations, intermittency, multistability, chaos – in other words, rich dynamics that depend on the region in parameter space in which the system lives – are ubiquitous features of coupled or forced nonlinear oscillators. Thus, in the present article, the nonlinear oscillator, exemplified by data obtained from studies of human rhythmical behavior, is the chief target of study.

Of course, this level-independent strategy on complex systems means that one must be able to place the system under study in a context that will enable one to deduce lawful structure from real data that are often noisy, the so-called inverse problem. Above, we mentioned using phase transitions or behavioral instabilities as a special entry point where laws may be found. Instabilities serve to demarcate behavioral patterns, thereby allowing a precise identification of order parameters for patterns and their nonlinear dynamics [5,19,20]. But what other methods may allow one to obtain dynamical laws (equations of motion) directly from measurements? And what steps can be taken to optimize these descriptions? Many tools and techniques have been employed in efforts to characterize the dynamics of complex systems, includ-

ing dimension, metric entropy, Lyapunov exponents and so forth (see e.g. [6]). Here we introduce a new method that reconstructs the entire flow vector field from the individual trajectory of the system, i.e. the evolution of a point in a low dimensional state space. The reconstructed flow vector or ordinary differential equation constitutes a global description, i.e. is valid on the whole subspace of the state space where the experimental data reside. We discuss the advantages and disadvantages of this method in the article itself.

Reconstruction of the underlying dynamics, of course, does not necessarily provide a physical explanation or enhance scientific insight, per se. Nevertheless, when complemented by theoretically motivated experimental observations it may still be possible to connect model equations to deeper levels of description. Certainly, in the history of science there are examples where an initially guessed mathematical formalism and eventual physical interpretation led to the same results. In the case of complex systems, where it is often not possible to derive a mathematical structure from first principles, these reconstruction methods could play an important role.

One possible application for methods of reconstruction is in the field of controlling dynamical systems. It may be possible, for instance, to control very complex systems if the dynamics of the system can be described with a small number of order parameters. Of course, for such control it is necessary to describe the dynamics of the uncontrolled system as exactly as possible. Here reconstruction can be an important tool.

The goal of the present work is not simply to analyze experimental material, but to show, in general, what is or might be possible in the field of reconstruction from time series. The described methods can be used in all fields of scientific research, where dynamical variables are observed. Biological movements, the dynamics of electrical potentials in nerves, fluid mechanics or the dynamics of chemical reactions are some examples. The central ansatz of the method is simply that any observations are produced by a dynamical system in the presence of noise. The aim is to provide a minimal model that reproduces observed behavior. We hope that the given examples will help researchers to think about possible applications of the present methods in their field of interest.

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modeltype	l_{max}	$c_{0,1}$	$c_{1,0}$	$c_{2,1}$	other c
ODE	4	1.147	−0.9910	−1.041	< 0.065
ODE	5	1.129	−0.9729	−1.028	< 0.042
ODE	6	1.135	−0.9564	−1.024	< 0.035
ODE	7	1.128	−0.9472	−1.021	< 0.030
map	4	1.199	−1.102	−1.157	< 0.019
map	5	1.015	−0.9321	−0.9859	< 0.030
map	6	1.006	−0.9296	−0.9802	< 0.036
map	8	1.009	−0.9170	−0.9781	< 0.041
map	9	1.013	−0.9175	−0.9795	< 0.039

Table 1: The coefficients for the fit of a simulated van der Pol oscillator (eq. 3.1). τ was 0.067 for fit with an ODE and 0.02 for map. The other parameters were: $b = 7$, $j_{max} = 100$, the Δt_l were chosen according to eq.(2.6). The used data were without noise and without transient. Map coefficients are rescaled according to eq.(2.10)

l_{max}	$c_{0,1}$	$c_{1,0}$	$c_{2,1}$	other c
4	1.594	−1.230	−1.125	< 0.317
5	1.395	−1.476	−1.103	< 0.308
6	1.153	−1.293	−0.9650	< 0.194
7	0.9140	−0.8094	−0.8282	< 0.045

Table 2: The coefficients for the fit of a simulated van der Pol oscillator (eq. 3.1) with an ODE. The time series was with noise (see fig.1 a) and with transient data. The other parameters were: $\tau = 0.067$, $b = 7$, $j_{max} = 100$, the Δt_l were chosen according to eq.(2.6).

coefficient	$l_{max} = 4$	$l_{max} = 4$	$l_{max} = 5$	$l_{max} = 6$	$l_{max} = 7$
$c_{0,0}$	0.306	-0.022	-0.433	-0.089	-0.232
$c_{0,1}$	4.063	3.173	4.349	4.235	3.814
$c_{0,2}$	-0.117	-0.034	0.102	0.006	0.043
$c_{0,3}$	-0.638	-0.455	-0.708	-0.693	-0.621
$c_{1,0}$	-1.338	-1.708	-0.868	-0.391	-0.529
$c_{1,1}$	0.131	0.009	-0.058	-0.045	0.003
$c_{1,2}$	0.990	0.829	0.943	0.750	0.662
$c_{2,0}$	-0.046	0.010	-0.133	0.061	0.092
$c_{2,1}$	-2.089	-1.883	-2.054	-1.878	-1.750
$c_{3,0}$	-0.022	0.104	-0.137	-0.237	-0.187

Table 3: The coefficients for the fit of a simulated van der Pol oscillator (eq. 3.1), as in table (2) but without transient data. The second column with $l_{max} = 4$ is the result of a fit with $j_{max} = 200$. Note the large differences in the linear coefficient $c_{1,0}$.

coefficient	short time s.	SD	long time s.	SD	$2l\tau$	flow method
$c_{0,0}$	1.227	0.528	-1.169	0.198	-0.856	-1.368
$c_{0,1}$	0.916	0.092	0.682	0.219	0.729	0.424
$c_{0,2}$	-0.599	0.132	-0.008	0.029	-0.075	0.055
$c_{0,3}$	-0.189	0.032	-0.064	0.037	-0.077	-0.009
$c_{1,0}$	-4.825	0.366	-4.359	0.246	-5.524	-3.856
$c_{1,1}$	0.849	0.040	-0.827	0.068	1.019	0.800
$c_{1,2}$	0.063	0.061	0.074	0.065	0.263	0.034
$c_{2,0}$	-1.094	0.399	0.897	0.138	0.875	0.976
$c_{2,1}$	-1.083	0.107	-0.615	0.150	-0.760	-0.448
$c_{3,0}$	0.452	0.127	0.670	0.147	1.488	0.324

Table 4: The coefficients for the fit of an experimental time series, a single hand movement as in fig.(2) with the trajectory method. The standard deviation was calculated with the coefficients from fits with 6 different fitparameters in the case of the short time series and 9 different fitparameters in the case of the long time series. For these fits the Δt_l were chosen according to eq.(2.6). Some of the coefficients are small and can be neglected, $c_{1,2}$ in both cases and $c_{0,2}$ for the long time series. The cubic term in the velocity $c_{0,3}$ is small but important for the stability of the model equations. Furthermore the result with Δt_l chosen according to eq.(2.7) is listed. For comparison, the result of a fit with the flow method is listed, the difference is basically in the cubic term in the velocity $c_{0,3}$, which is very small.

Figure 1: (a) A trajectory of a simulated van der Pol oscillator (eq. 3.1) with noise in a state space with position and velocity as state vector components. The trajectory up to the point marked with t was excluded for fits without transient data.
(b) A trajectory in the same state space for the model equation of table (3) with $l_{max} = 7$. Although the coefficients are very different from the coefficients of eq.(3.1) the shape of the trajectory simulated with the model equation is very similar to the trajectory of the van der Pol oscillator.

Figure 2: (a) A trajectory of an experimental single hand movement (around wrist) at a frequency of 3Hz in arbitrary units in a state space with position and velocity as state vector components. Trajectories in the same state space for the model equations of table (4), (b) long time series, (c) short time series, (d) flow method.

Figure 3: (a) Displacement for a single finger movement at a frequency of 1.5Hz in arbitrary units. Note the pause at maximal displacement. (b) Time series of the model equation obtained by the flow method. (c) Same time series as (b) in state space with position and velocity as state vector components.

Figure 4: (a) (b) Trajectory of eq.(4) in the state space given by (2). The slightly distorted line is the simulated time series with noise, the straight line is a simulation of the model equation with the same initial condition. For ease of visualization in (c) and (d) we plot the positions of both oscillators against each other to show the transition from anti-phase to in-phase pattern. (c) is a simulation of eq.(4) (without noise) and (d) is a trajectory of the model equation obtained by the trajectory method.

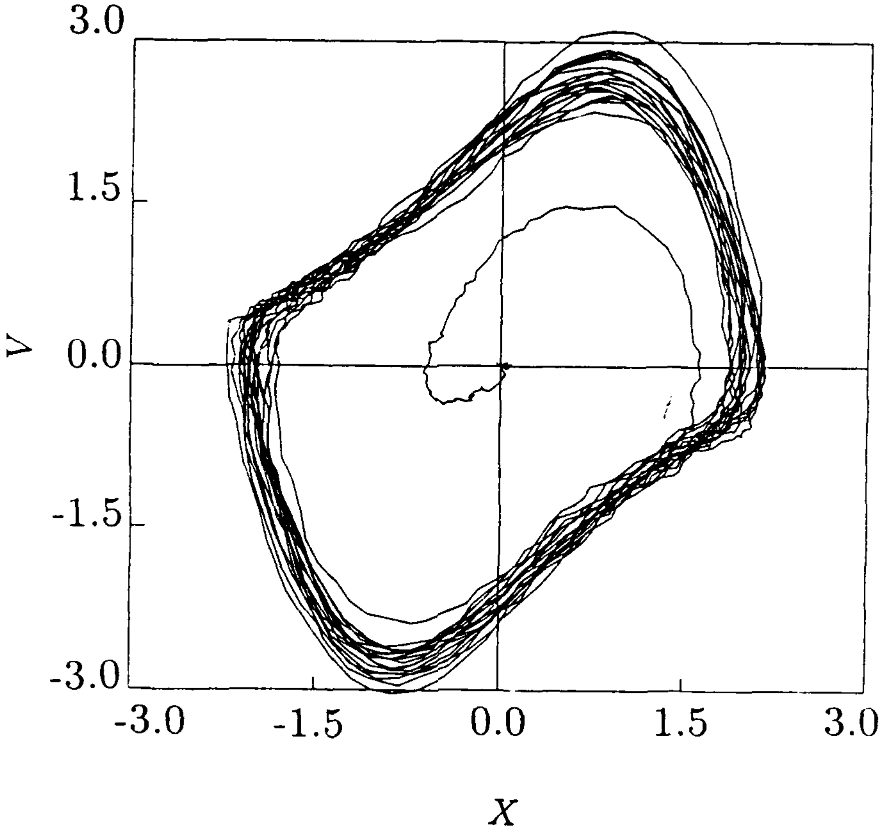
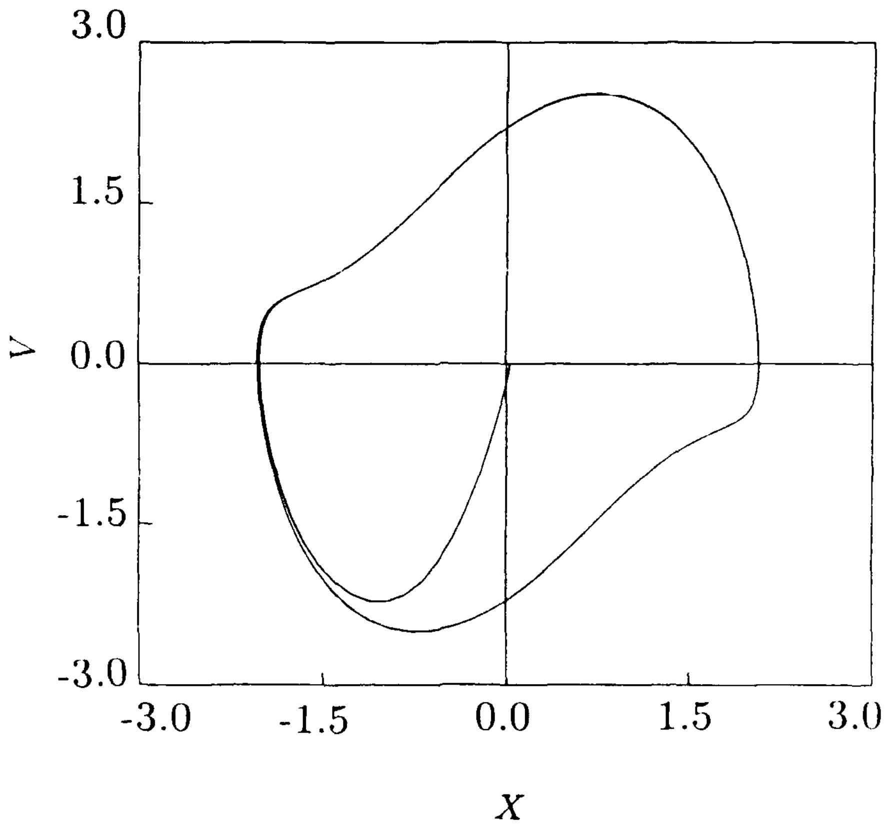


Figure 1(a)



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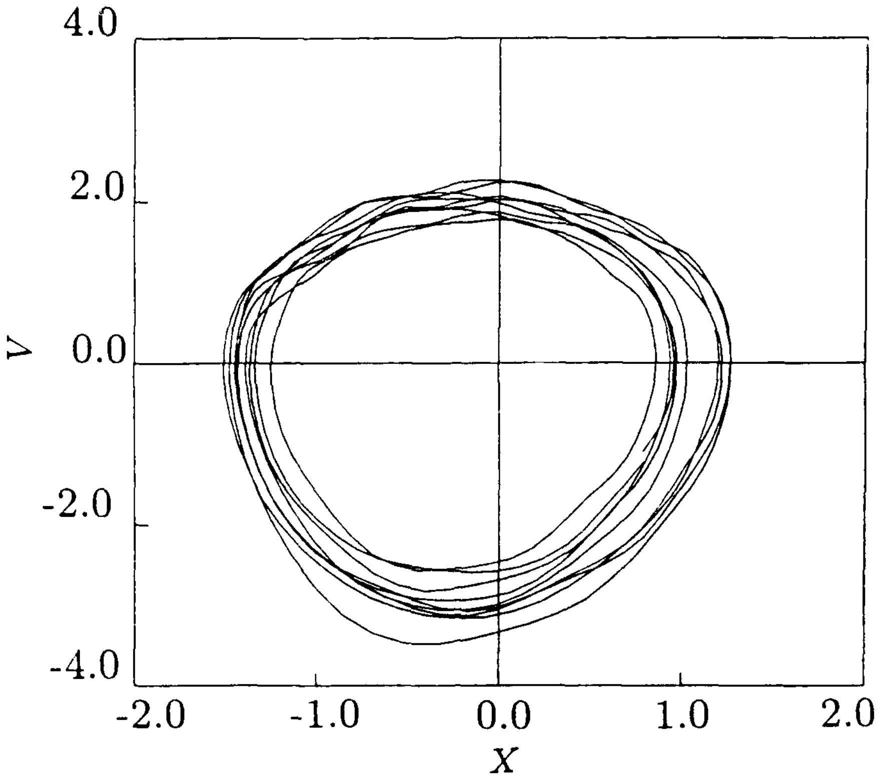
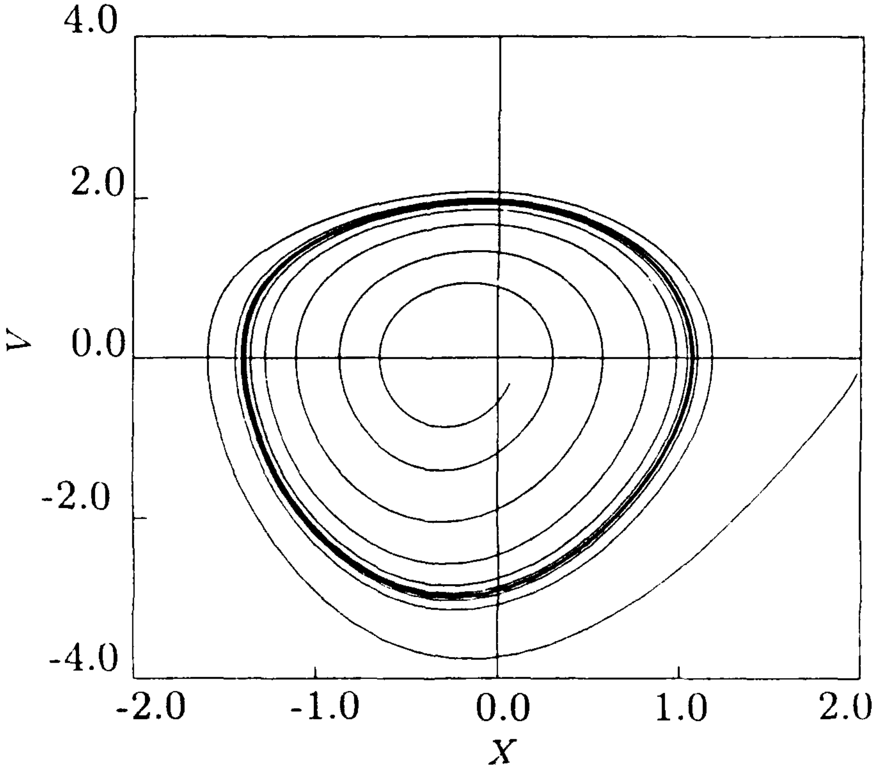


Figure 2(a)



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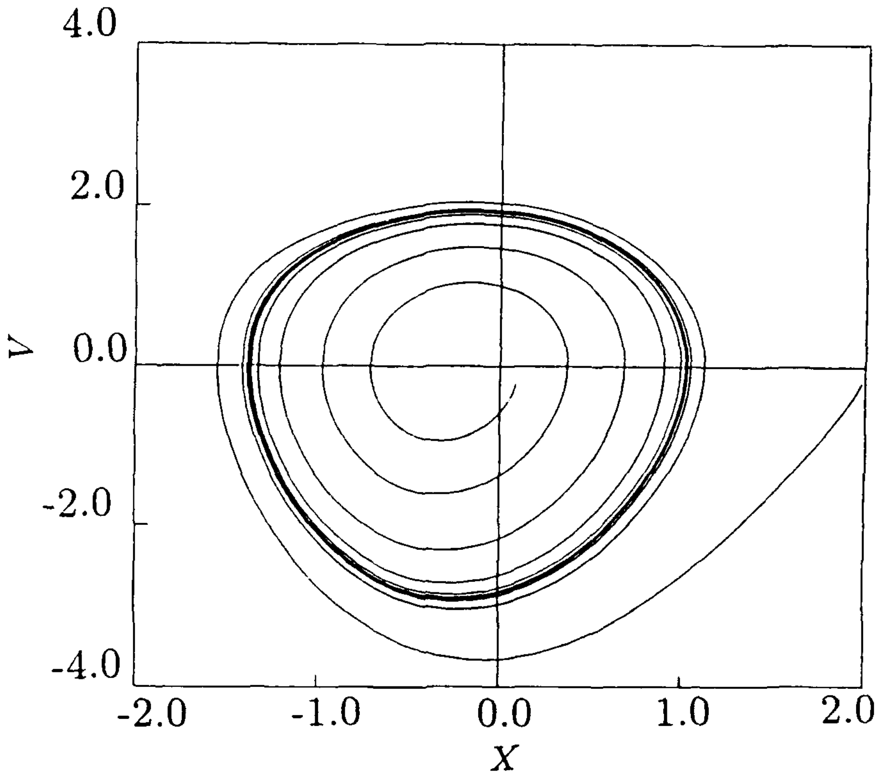

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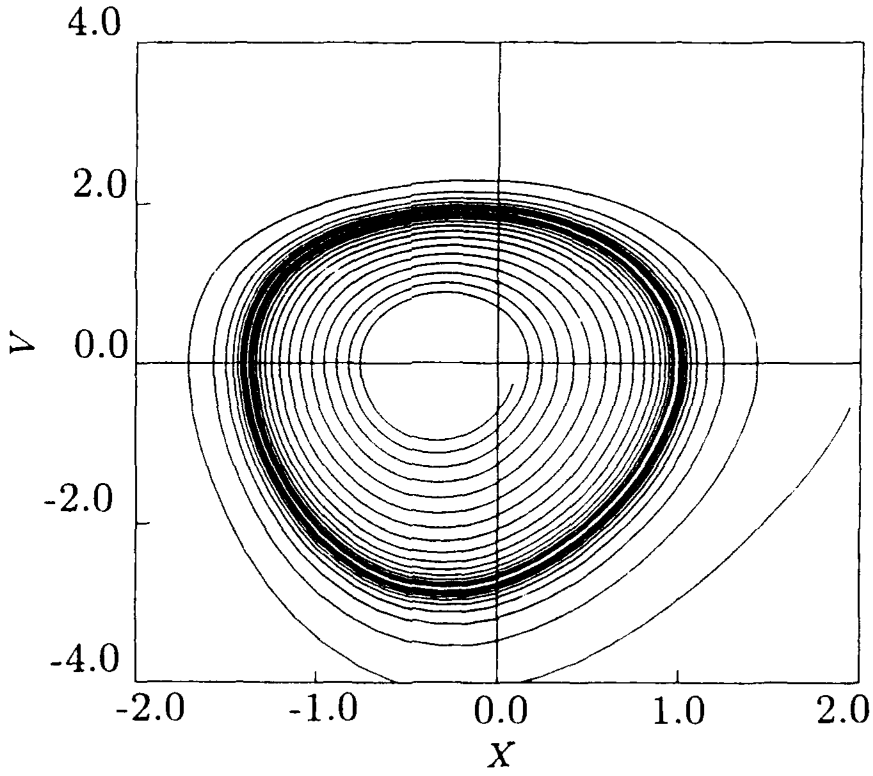

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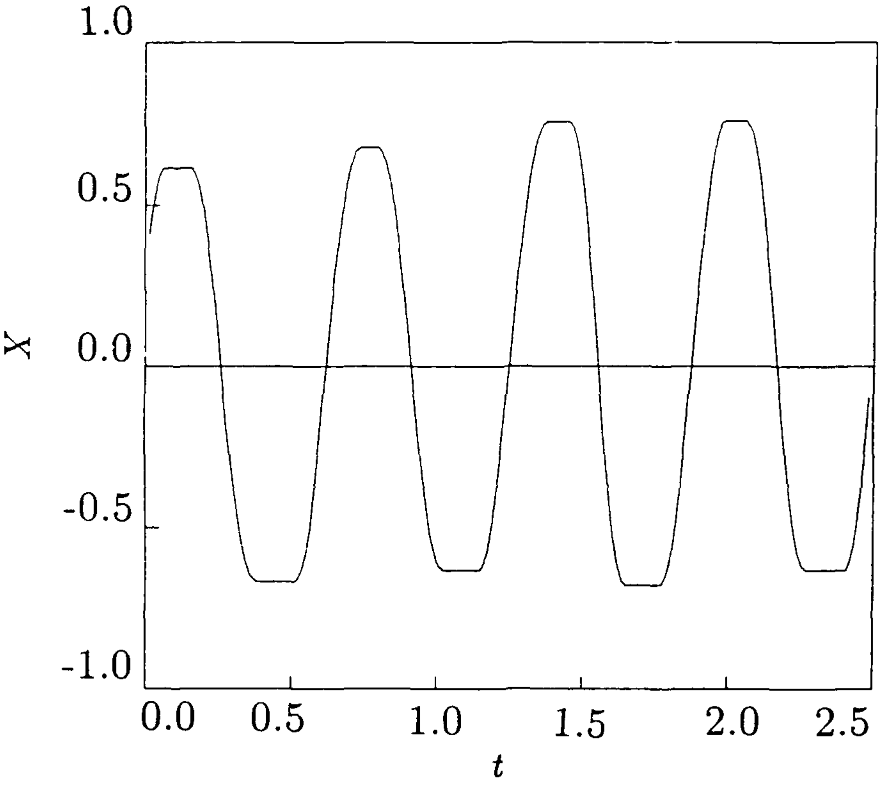
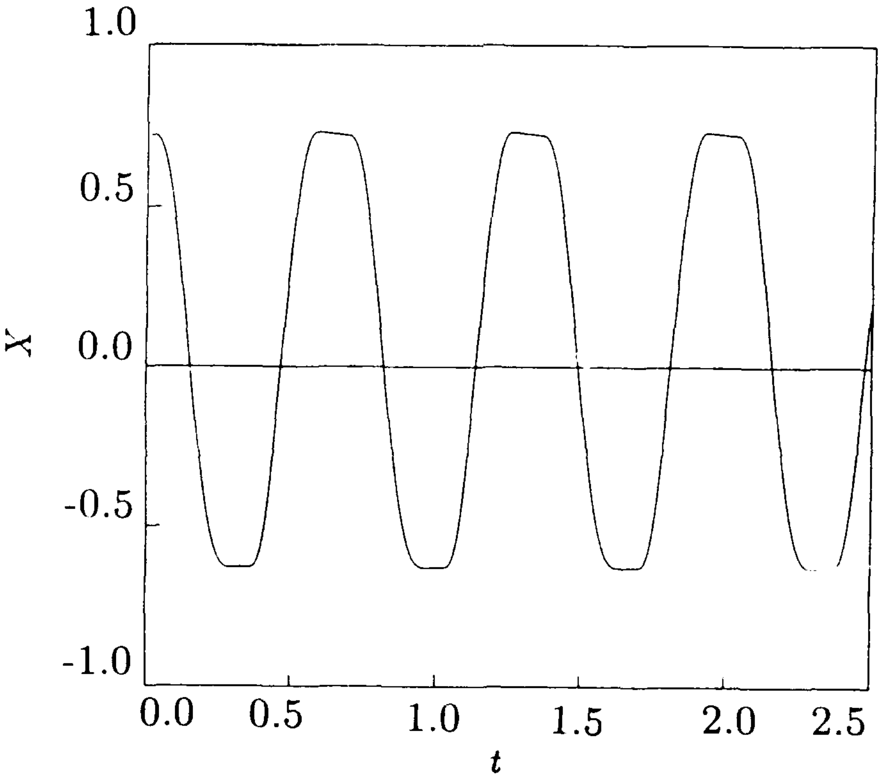


Figure 3(a)



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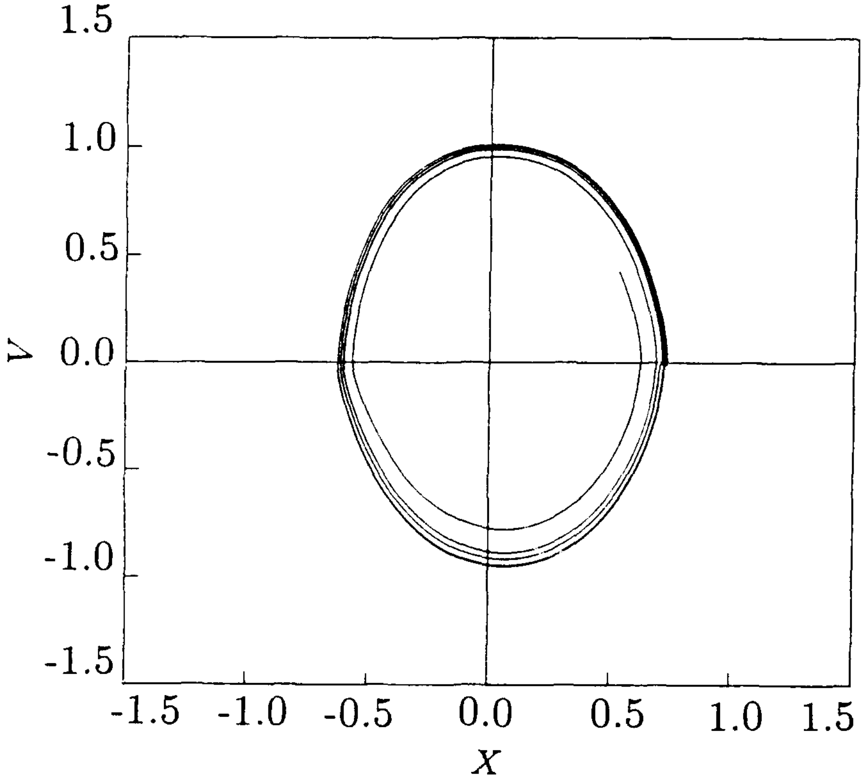

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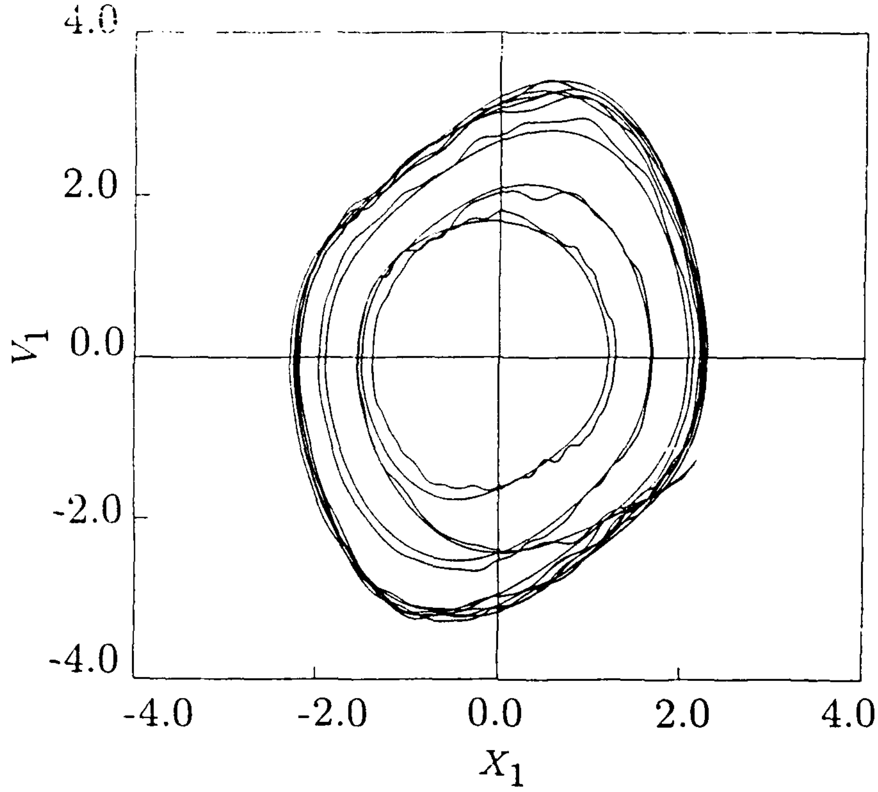
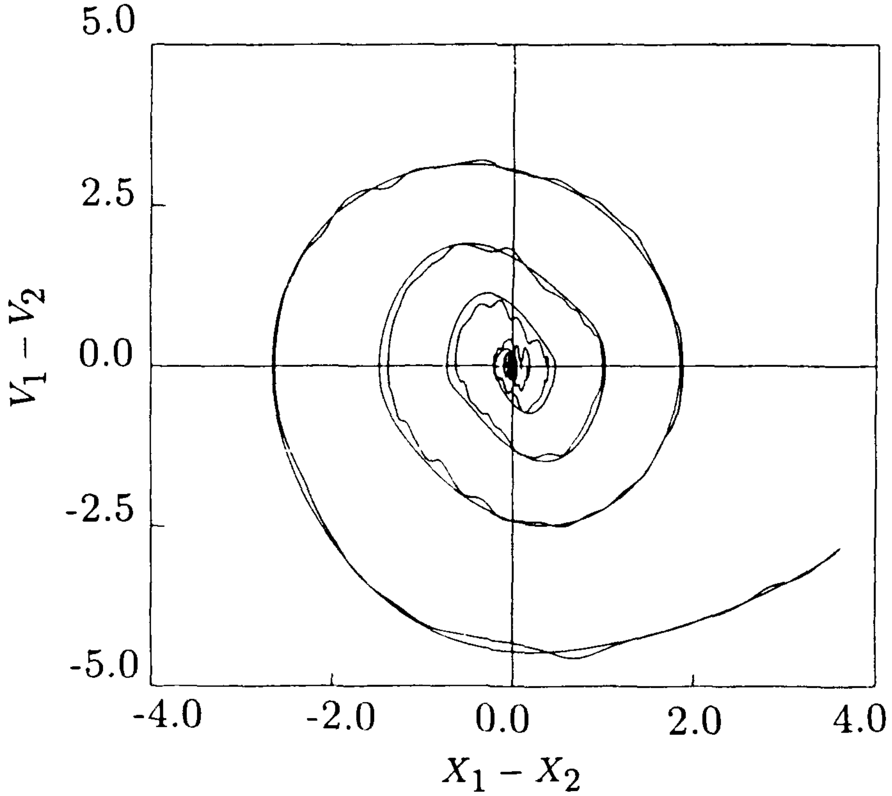


Figure 4(a)



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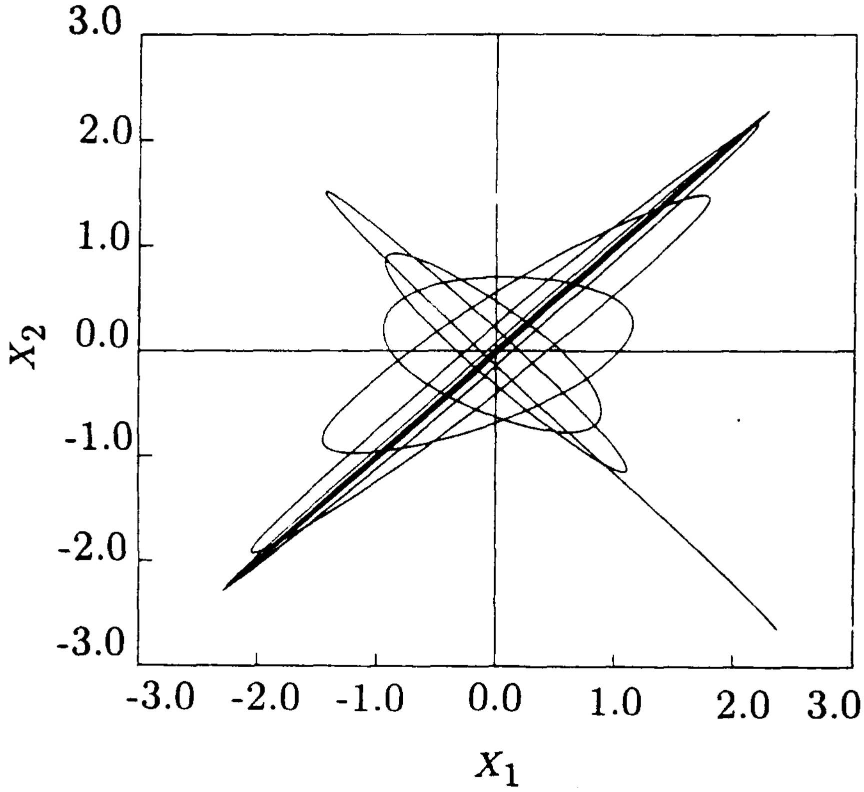

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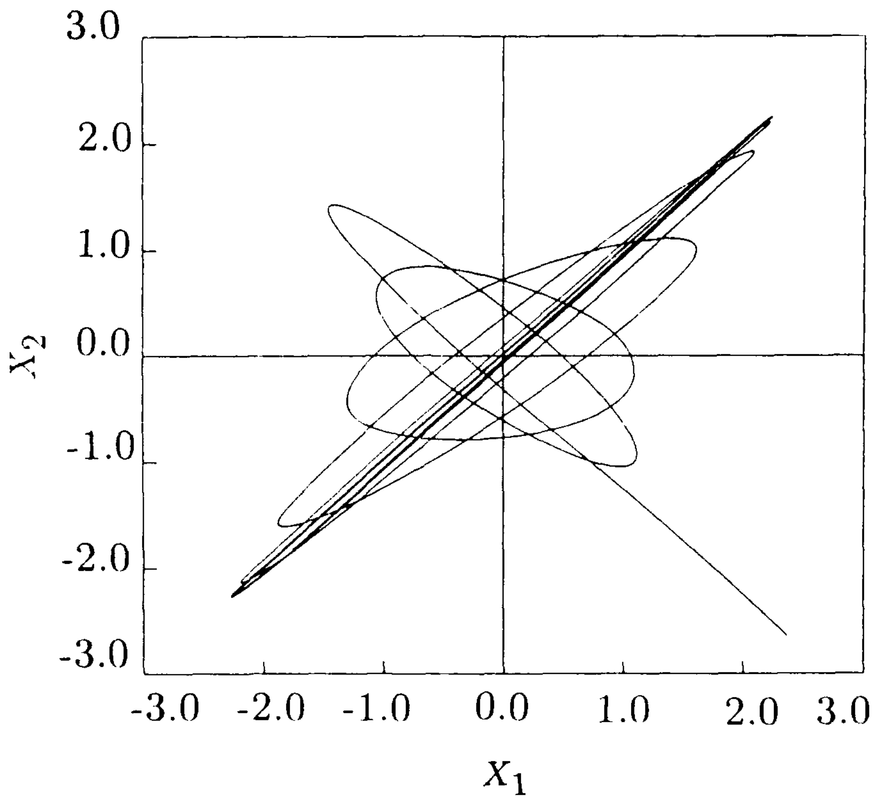

Figure 4(d)